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MULTIPLE JOB TRANSITIONS

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Product forms for queueing networks with state dependent multiple job transitions

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Abstract

A general framework of continuous-time queueing networks is studied with simultaneous state dependent service completions such as due to concurrent servicing or discrete-time slotting and with state dependent batch routings such as most typically modelling blocking. By using a key notion of group-local-balance necessary and sufficient conditions are given for the stationary distribution to be of product form. These conditions and a constructive computation of the product form are based upon merely local solutions of the group-local-balance equations which can usually be solved explicitly for concrete networks. Moreover, a decomposition theorem is presented to separate service and routing conditions. General batch service and batch routing examples yielding a product form are hereby concluded. As illustrated by various examples known results on both discrete- and continuous-time queueing networks are unified and extended.

Keywords: Queueing networks, product form, discrete-time queueing network, batch service, batch routing, blocking, group-local-balance.

Contents

1 Introduction

Ever since Jacksons classical product form results (cf. [9], [10]), queueing networks have been extensively studied. Most notably, among various others, necessary and sufficient conditions for stationary product form distributions and related insensitivity results have been provided in [1], [3], [4], [5], [8], [11], [12], [13], [14], [17], [18], [20], [25], [26]. These references all concern queueing networks in which changes occur due to one job at a time, such as an external arrival, a transfer from one station to another or a system departure.

Typical present-day applications, however, feature simultaneous job transitions. For example, in manufacturing, parts are often processed and transported in batches, in parallel programming a number of modules can be initiated and run at the same time and, most notably, in digitized or time-slotted communication networks (e.g. ISDN), messages or packets are simultaneously transmitted and released at discrete times.

A number of product form extensions to queueing networks with batch servicing have been established over the last couple of years. In [6] product form results are derived for open discrete-time Jacksonian networks and in [19] for a closed queueing model of a computer system both with so called doubly stochastic disciplines under the conditions that at any station no more than one job can either arrive or leave at the same time. In these references generally distributed services are allowed. Similar results were independently obtained in [24] for open discrete-time Jackson networks with geometric services as based upon the concept of quasi-reversibility and in [15], [16] for discrete-time Jackson networks with various discrete-time service initiating and finishing protocols. In a recent paper [21] these results were generalized to closed networks with a generalized form of a total system dependent batch servicing while jobs route independently. Herein, a key-concept of balance per group is introduced as the responsible factor for product form results. In [7] an extension of

these results is given to mixed open and closed networks with batch job routing. All these references so far do not include blocking. Recently, in [2] a related framework has been presented in which under the restriction of a reversibility assumption not only state dependent multiple job servicing but also state dependent multiple job routing and blocking are allowed.
The present paper investigates the possible extensions to general state dependent simultaneous servicing and routing such as most notably modelling blocking phenomena also when a non-reversible batch routing is involved. The model is thus more general than in [2] as non-reversible structures are allowed, but more restrictive as it particularizes to queueing networks and therefore does not cover other multi-transitional models such as clustering processes. Further detailed comparison with the preceding literature will be given later on.
Based upon the key-notion of group-local-balance introduced in [21] and a technique of defining artificial Markov chains as introduced in [8] for the case of single changes, we obtain the following novel results:

- A characterization of closed form expressions (e.g. of product form) by means of local solutions of restricted local balance equations;
- A constructive method for computing these expressions related to Kolmogorov's criterion;
- A number of new product form examples of queueing networks with batch servicing and routing such as with batch dependent and independent job routing. Particular examples are given with:
  - non-reversible structures and blocking,
  - routing selective blocking,
  - limited clusters of stations.

The outline of this paper is as follows. In section 2 we introduce group-local-balance and local state spaces and give our main result stating that the stationary distribution of the queueing network can be computed by local solutions if group-local-balance holds. In section 3 we first present a decomposition theorem by which service and routing conditions can be analyzed separately. Next we give some general applications to queueing networks allowing batch services and batch routing with state dependent blocking. Most notably, discrete-time servicing protocols and non-reversible batch routings with blocking are hereby included. In section 4 we give some explicit examples:

- Discrete-time networks without blocking;
- Cyclic queueing networks with batch servicing and blocking;
- Queueing networks with cluster dependent simultaneous services.

2 Model

We consider a continuous-time queueing network consisting of N stations, labeled $1, 2, \ldots, N$ and restrict our discussion to one type of customers or jobs that move between the stations.

We assume that the queueing network can be represented by a continuous-time Markov chain with state space S. A state $\bar{n} = (n_1, \ldots, n_N)$ is a vector with components n_i, $i = 1, \ldots, N$ where n_i denotes the number of customers or jobs at station i, $i = 1, \ldots, N$. The transition rate from state $\bar{n}$ to state $\bar{n}'$ is denoted by $q(\bar{n}, \bar{n}')$. First we discuss the form of the transitions as multiple changes are allowed. Consider a transition from state $\bar{n}$ to state $\bar{n}'$ due to g_i jobs that leave station i, g_i' jobs that enter station i and m_i jobs that remain at station i, $i = 1, \ldots, N$. We can thus write $\bar{n} = \bar{m} + \bar{g}$ and $\bar{n}' = \bar{m} + \bar{g}'$ where $\bar{m} = (m_1, \ldots, m_N)$ is the vector of remaining jobs while $\bar{g} = (g_1, \ldots, g_N)$ and $\bar{g}' = (g_1', \ldots, g_N')$ are the vectors representing departing and entering jobs respectively. The corresponding transition rate for this particular transition is denoted by $q(\bar{g}, \bar{g}'; \bar{m})$. Note that for given groups $\bar{g}$, $\bar{g}'$ the decomposition of $\bar{n}$, $\bar{n}'$ in $\bar{n} = \bar{m} + \bar{g}$, $\bar{n}' = \bar{m} + \bar{g}'$ is unique, but that a transition from state $\bar{n}$ to state $\bar{n}'$ might also occur due to other groups. The total transition rate from state $\bar{n}$ to state $\bar{n}'$ is then given by

$$q(\bar{n}, \bar{n}') = \sum_{\{\bar{g}, \bar{g}', \bar{m}:\ \bar{m}+\bar{g}=\bar{n},\ \bar{m}+\bar{g}'=\bar{n}'\}} q(\bar{g}, \bar{g}'; \bar{m}). \tag{2.1}$$

Remark 2.1 (Transitions) We do not exclude $g_i > 0$ and $g_i' > 0$ in which case the total number of jobs at station i before and after the transition is at least $m_i + 1$. Physically, this also allows jobs to leave and enter the same station. Also for $\bar{g} \neq 0$ we can have $q(\bar{g}, \bar{g}; \bar{m}) \neq 0$ which we refer to as a dummy transition. □

Remark 2.2 (Open model) Open models are included in our formulation. When $\sum_{i=1}^N g_i > \sum_{i=1}^N g_i'$ jobs depart from the system and when $\sum_{i=1}^N g_i < \sum_{i=1}^N g_i'$ jobs arrive at the system. Note that also in the case $\sum_{i=1}^N g_i = \sum_{i=1}^N g_i'$ jobs may enter and leave the system, but in this case the total number of jobs in the queueing network remains the same. □

Remark 2.3 (Fixed routing groups) Notice that in a transition from state $\bar{n}$ to state $\bar{n}'$ the routing groups $\bar{g}$ and $\bar{g}'$ are completely determined if the vector $\bar{m}$ is fixed. □

First let us briefly illustrate our notation.

Example 2.4 (Notation) Consider the closed cyclic three station queueing network as depicted in Figure 1. We will discuss several types of transitions so as to illustrate our notation. Throughout this example we will assume that in the initial state there are 4 jobs at station 1, 3 jobs at station 2 and 3 jobs at station 3, i.e. $\bar{n} = (4, 3, 3)$. If 2 jobs leave station 3 then they must route to station 1. For this particular transition we have

$$
\begin{aligned}
\bar{n} &= (4,3,3) \\
\bar{m} &= (4,3,1) \\
\bar{g} &= (0,0,2) \\
\bar{g}' &= (2,0,0) \\
\bar{n}' &= (6,3,1).
\end{aligned}
$$

Now consider a transition in which 2 jobs leave station 1 and 2 jobs leave station 2. Then we have

$$
\begin{aligned}
\bar{n} &= (4,3,3) \\
\bar{m} &= (2,1,3) \\
\bar{g} &= (2,2,0) \\
\bar{g}' &= (0,2,2) \\
\bar{n}' &= (2,3,5).
\end{aligned}
$$

Notice that in this transition $n_2 = n'_2 = 3$ although there are jobs leaving and entering station 2. In fact it seems that the 2 routing jobs route from station 1 to station 3 without visiting station 2. This turns out to be an important feature of batch service queueing networks as it will influence blocking protocols (see section 4). A transition in which for example 2 jobs leave station 1, 2 jobs leave station 2 and 2 jobs leave station 3 is a dummy transition. In this case

$$
\begin{aligned}
\bar{n} &= (4,3,3) \\
\bar{m} &= (2,1,1) \\
\bar{g} &= (2,2,2) \\
\bar{g}' &= (2,2,2) \\
\bar{n}' &= (4,3,3).
\end{aligned}
$$

Although there are jobs routing among the stations the state of the network does not change. Notice that this is a feature of batch service models only. Further notice that since the state of the network does not change in this type of transition we do not have to include these dummy transitions in the balance equations. Essentially there is no difference between the dummy transition illustrated above and a standard "transition" in which no jobs move at all. □

We make the following assumptions.

(A1) The Markov chain is irreducible for a set $V \subset S$ and there exists a unique stationary distribution π at V.

(A2) The total transition rate out of each state is uniformly bounded, i.e. for all $\bar{n} \in V$ and some constant C

$$q(\bar{n}) = \sum_{\bar{n}' \neq \bar{n}} q(\bar{n}, \bar{n}') \leq C < \infty.$$

In the following Definition 2.5 we define a type of partial balance which plays a key-role throughout this paper. First we give a motivation for this definition. Based upon (A1) and (A2) a distribution π at V is stationary if and only if for all $\bar{n} \in V$ (cf. [11])

$$\pi(\bar{n}) \sum_{\bar{n}' \neq \bar{n}} q(\bar{n}, \bar{n}') = \sum_{\bar{n}' \neq \bar{n}} \pi(\bar{n}') q(\bar{n}', \bar{n}). \tag{2.2}$$

Inserting (2.1) into (2.2) and rearranging summations yields that a distribution π at V is stationary if for all $\bar{n} \in V$

$$\pi(\bar{n}) \sum_{\{\bar{g}, \bar{m}: \bar{m}+\bar{g}=\bar{n}\}} \sum_{\bar{g}' \neq \bar{g}} q(\bar{g}, \bar{g}'; \bar{m}) = \sum_{\{\bar{g}, \bar{m}: \bar{m}+\bar{g}=\bar{n}\}} \sum_{\bar{g}' \neq \bar{g}} \pi(\bar{m} + \bar{g}') q(\bar{g}', \bar{g}; \bar{m}). \tag{2.3}$$

Definition 2.5 (Group-local-balance property) *A distribution p at an irreducible set V satisfies group-local-balance (GLB) at V if for all $\bar{m}$ and $\bar{m} + \bar{g} \in V$*

$$p(\bar{m} + \bar{g}) \sum_{\bar{g}' \neq \bar{g}} q(\bar{g}, \bar{g}'; \bar{m}) = \sum_{\bar{g}' \neq \bar{g}} p(\bar{m} + \bar{g}') q(\bar{g}', \bar{g}; \bar{m}). \tag{2.4}$$

A Markov chain has the GLB-property if the stationary distribution satisfies (2.4).

Remark 2.6 (Reference [8]) The definition can be seen as a generalization of notions as local balance (cf. [17]) or job-local-balance (cf. [8]) for the case of single changes (i.e. only one job is allowed to leave a station). The approach that we will follow from here on is a generalization to multiple changes of the approach followed in [8]. □

From (2.3) it is clear that a distribution satisfying GLB is stationary. In general however (2.4) is much more restrictive than (2.2) and does not need to have a solution. First we analyze the feasibility of (2.4).
For fixed $\bar{m}$ let $V(\bar{m})$ be the state space of the Markov chain with transition rates $q(\bar{g}, \bar{g}'; \bar{m})$ restricted to V. Further let $V_i(\bar{m})$, $i = 1, \dots, k(\bar{m})$ denote the irreducible sets in $V(\bar{m})$ with respect to the Markov chain with transition rates $q(\bar{g}, \bar{g}'; \bar{m})$. We will refer to $V(\bar{m})$ as local state space and to $V_i(\bar{m})$ as the local subsets.

Remark 2.7 (Local state space) Note that a state $\bar{n}$ may be an element of different local state spaces $V(\bar{m})$. This allows transitions from one local state space to another since in general for $\bar{m} \neq \bar{m}'$ we have $V(\bar{m}) \cap V(\bar{m}') \neq \emptyset$. □

Example 2.8 (GLB and irreducible sets) Reconsider the cyclic three center model of Figure 1 in which 10 jobs are present. If $\bar{m} = (4, 3, 1)$ there are 2 jobs routing among the stations. If all transitions are possible the local state space is

$$V(4,3,1) = \{(5,4,1), (4,4,2), (5,3,2), (6,3,1), (4,5,1), (4,3,3)\}.$$

In the first three states there is one departing job at two stations. In the remaining three states two jobs depart from the same station. The queueing network is cyclic so we can split up the local state space into two irreducible sets, one local subset for two jobs departing from different stations and one local subset for two jobs departing from the same station

$$V_1(4,3,1) = \{(5,4,1), (4,4,2), (5,3,2)\}$$
$$V_2(4,3,1) = \{(6,3,1), (4,5,1), (4,3,3)\}.$$

As will be seen in section 4 the notion of different irreducible sets V_i is important when blocking occurs. □

In the following lemma we show that $V(\bar{m})$ consists of irreducible sets only if the Markov chain satisfies GLB.

Lemma 2.9 *If the stationary distribution π satisfies GLB, then for any $\bar{m}$*

$$V(\bar{m}) = \bigcup_{i=1}^{k(\bar{m})} V_i(\bar{m}). \tag{2.5}$$

Proof Without loss of generality we may set $V(\bar{m}) \neq \emptyset$. We only need to prove that the Markov chain at $V(\bar{m})$, for fixed $\bar{m}$, has no transient states. By virtue of the GLB-property, however, this local Markov chain has a stationary distribution $c\,\pi$ where

$$c^{-1} = \sum_{\{\bar{g}:\bar{m}+\bar{g}\in V\}} \pi(\bar{m}+\bar{g}).$$

As $\pi(\cdot) > 0$ each state is positive recurrent. □

From Lemma 2.9 it follows that, if GLB holds, then for any fixed $\bar{m}$ for which $V(\bar{m}) \neq \emptyset$ and $i \in \{1,\ldots,k(\bar{m})\}$ the following set of equations has a unique positive solution up to a constant factor.

$$x(\bar{g};\bar{m}) \sum_{\bar{g}'\neq\bar{g}} q(\bar{g},\bar{g}';\bar{m}) = \sum_{\bar{g}'\neq\bar{g}} x(\bar{g}';\bar{m})q(\bar{g}',\bar{g};\bar{m}) \qquad \bar{m}+\bar{g} \in V_i(\bar{m}) \tag{2.6}$$

We now make the following essential assumption in order to satisfy GLB. This assumption is justified by Lemma 2.9.

(A3) For any fixed $\bar{m}$ for which $V(\bar{m}) \neq \emptyset$ relation (2.5) holds and for $i \in \{1,\ldots,k(\bar{m})\}$ the system (2.6) has a unique positive solution $\{x(\bar{g};\bar{m}) | \bar{m}+\bar{g} \in V_i(\bar{m})\}$ up to a constant factor.

Remark 2.10 **(Singletons)** If $V(\bar{m}) \neq \emptyset$ but $q(\bar{g},\bar{g}';\bar{m}) = 0$ for all $\bar{g},\bar{g}'$ then $V(\bar{m})$ consists of singletons only, i.e. for $i = 1,\ldots,k(\bar{m})$ the local subsets consist of one state only. In this case relation (2.6) is trivially fulfilled. □

The equations (2.6) can often be solved explicitly. We therefore intend to give a characterization and computation of the stationary distribution which satisfies GLB purely in terms of these local solutions of the global balance equations. To this end, we first define an additional process with transition rates $\bar{q}$.

Definition 2.11 **($\bar{q}$-process)** *A Markov chain at V with transition rates $\bar{q}$ which satisfy the following relations is called a $\bar{q}$-process.*

$$
\begin{cases}
\textit{For any } \bar{m},\ i = 1, \ldots, k(\bar{m}) \textit{ and } \bar{m} + \bar{g},\ \bar{m} + \bar{g}' \in V_i(\bar{m}) \\
\qquad \dfrac{\bar{q}(\bar{g}, \bar{g}'; \bar{m})}{\bar{q}(\bar{g}', \bar{g}; \bar{m})} = \dfrac{x(\bar{g}'; \bar{m})}{x(\bar{g}; \bar{m})} \\
\textit{and otherwise} \\
\qquad \bar{q}(\bar{g}, \bar{g}'; \bar{m}) = 0.
\end{cases}
\tag{2.7}
$$

Remark 2.12 ($\bar{q}$-process) Note that the transition rates $\bar{q}$ are uniquely defined by (2.7) up to a common factor at each of the irreducible local subsets $V_i(\bar{m})$. The quotients of the $\bar{q}$ are thus unique. As will be apparent below we only need those quotients. Further notice that also for the $\bar{q}$-process a path $\bar{n}_0 \to \bar{n}_1 \to \cdots \to \bar{n}_{i-1} \to \bar{n}_i \to \bar{n}$ where $\bar{n}_0$, $\bar{n}$ are elements of different local state spaces is possible. This is a direct consequence of the fact already noticed in Remark 2.7. □

In Definition 2.11 we have defined the $\bar{q}$-process in terms of the local solutions $x(\bar{g}; \bar{m})$. These local solutions may be a function of $\bar{g}$ and $\bar{m}$ separately. The following definition relates these local solutions to a global solution $\bar{\pi}(\bar{m} + \bar{g})$ of the $\bar{q}$-process.

Definition 2.13 (Strong reversibility) *The $\bar{q}$-process is strongly reversible at V if for all $\bar{m}$ for which $V(\bar{m}) \neq \emptyset$ and $i \in \{1, \ldots, k(\bar{m})\}$ the stationary distribution $\bar{\pi}$ satisfies*

$$
\bar{\pi}(\bar{m} + \bar{g})\bar{q}(\bar{g}, \bar{g}'; \bar{m}) = \bar{\pi}(\bar{m} + \bar{g}')\bar{q}(\bar{g}', \bar{g}; \bar{m}) \quad \bar{m} + \bar{g}, \bar{m} + \bar{g}' \in V_i(\bar{m}). \tag{2.8}
$$

Remark 2.14 (Reversibility) Note that strong reversibility is defined at the local irreducible subsets, this in contrast with the standard notion of reversibility which is defined at the irreducible set V (cf. [11]). The relation between strong reversibility and standard reversibility is illustrated in Figure 2. A transition from state $\bar{n}$ to state $\bar{n}'$ may occur due to different routing groups. Suppose this transition is possible due to the departure of group $\bar{g}_1$ and arrival of group $\bar{g}_1'$ and due to the departure of group $\bar{g}_2$ and arrival of group $\bar{g}_2'$ as is illustrated in Figure 2. Then strong reversibility requires balance at the top path and the bottom path separately while standard reversibility requires balance on both paths simultaneously. It directly follows that strong reversibility is a much stronger form of balance than standard reversibility. □

Remark 2.15 (Single changes) If single changes are allowed only, reversibility and strong reversibility are equivalent. This is an immediate consequence of the uniqueness in each transition of the decomposition of $\bar{n}$ in $\bar{m}$ and $\bar{g}$ for single changes. □

We are now able to give our main theorem.

Theorem 2.16 *The stationary distribution π of a Markov chain at V satisfies GLB if and only if the $\bar{q}$-process is strongly reversible at V. Moreover with $\bar{\pi}$ its stationary distribution we have for all $\bar{n} \in V$*

$$\pi(\bar{n}) = \tilde{\pi}(\bar{n}). \tag{2.9}$$

Proof "if" If the $\bar{q}$-process is strongly reversible then from (2.8) and (2.7) it immediately follows that

$$\frac{\tilde{\pi}(\bar{m}+\bar{g})}{\tilde{\pi}(\bar{m}+\bar{g}')} = \frac{\bar{q}(\bar{g}',\bar{g};\bar{m})}{\bar{q}(\bar{g},\bar{g}';\bar{m})} = \frac{x(\bar{g};\bar{m})}{x(\bar{g}';\bar{m})}.$$

Hence, the distribution $\pi(\cdot) = \tilde{\pi}(\cdot)$ satisfies (2.6) which shows that π satisfies GLB. "only if" If π satisfies GLB then π is a solution of (2.6). Then, recalling (A3) we thus conclude

$$\frac{x(\bar{g};\bar{m})}{x(\bar{g}';\bar{m})} = \frac{\pi(\bar{m}+\bar{g})}{\pi(\bar{m}+\bar{g}')}$$

so that by (2.7) for any $i \in \{1,\ldots,k(\bar{m})\}$ and $\bar{m}+\bar{g}, \bar{m}+\bar{g}' \in V_i(\bar{m})$

$$\frac{\bar{q}(\bar{g},\bar{g}';\bar{m})}{\bar{q}(\bar{g}',\bar{g};\bar{m})} = \frac{\pi(\bar{m}+\bar{g})}{\pi(\bar{m}+\bar{g}')}. \tag{2.10}$$

As $\bar{q}(\bar{g},\bar{g}';\bar{m}) = \bar{q}(\bar{g}',\bar{g};\bar{m}) = 0$ if $\bar{m}+\bar{g}, \bar{m}+\bar{g}'$ are not contained in the same local subset $V_i(\bar{m})$ relation (2.8) implies that the $\bar{q}$-process is strongly reversible at V with $\tilde{\pi} = \pi$. □

Remark 2.17 (Reference [2]) In [2] reversible processes in which a transition from state $\bar{n}$ to state $\bar{n}'$ can occur due to exactly one group $\bar{g}$ are studied and necessary and sufficient conditions are given for the process to have a product form stationary distribution. These results remain valid when more than one group $\bar{g}$ can establish the transition from state $\bar{n}$ to state $\bar{n}'$ and reversibility is replaced by strong reversibility. The necessary and sufficient conditions from [2] can be applied in order to conclude that the the stationary distribution of the $\bar{q}$-process is of product form. Then by Theorem 2.16 it directly follows that this gives necessary and sufficient conditions for π to be of product form also. In this approach detailed information on the transition rates of the $\bar{q}$-process is needed. However, we will use a more direct approach (also see Remark 2.20). □

The following Corollary 2.18 shows the importance of the result of Theorem 2.16 as it enables us to check GLB and to conclude the corresponding stationary distribution based upon merely the local solution $x(\bar{g};\bar{m})$. These are often obtainable explicitly.

Corollary 2.18 *The stationary distribution π satisfies GLB if and only if for arbitrary reference state $\bar{n}_0$ and all $\bar{n} \in V$*

$$\prod_{k=0}^{p} \frac{\bar{q}(\bar{g}_k,\bar{g}_k';\bar{m}_k)}{\bar{q}(\bar{g}_k',\bar{g}_k;\bar{m}_k)} = c\pi(\bar{n}) \tag{2.11}$$

for all paths of the form

$$\begin{aligned} \bar{n}_0 = \bar{m}_0 + \bar{g}_0 \to \bar{m}_0 + \bar{g}_0' = \bar{m}_1 + \bar{g}_1 \to \bar{m}_1 + \bar{g}_1' = \cdots \\ \cdots = \bar{m}_p + \bar{g}_p \to \bar{m}_p + \bar{g}_p' = \bar{m}_{p+1} + \bar{g}_{p+1} = \bar{n} \end{aligned} \tag{2.12}$$

for which the denominator is positive and with arbitrary $p \in N$ and $c = \pi(\bar{n}_0)^{-1}$ denoting a normalizing constant.

Proof The proof below is very similar to that of the Kolmogorov criterion for reversibility, but since in the Kolmogorov criterion reversible processes are used, for completeness, we shall give it here for strongly reversible processes.
"if" By virtue of Theorem 2.16 it is sufficient to prove strong reversibility for the $\bar{q}$-process. To this end, consider some arbitrary i, $\bar{g}, \bar{g}', \bar{m}$ such that $\bar{n} = \bar{m} + \bar{g}, \bar{n}' = \bar{m} + \bar{g}' \in V_i(\bar{m})$, and an arbitrary reference state $\bar{n}_0$. By the irreducibility of V there exists a path $\bar{n}_0 \to \bar{n}$ which is of the form stated above in (2.12). Then, by virtue of (2.7) we also have a path $\bar{n}_0 \to \bar{n}'$ of the form $\bar{n}_0 \to \bar{n} = \bar{m} + \bar{g} \to \bar{n}' = \bar{m} + \bar{g}'$ such that the denominator in the next expression, as according to (2.11), is positive

$$\frac{\pi(\bar{n}')}{\pi(\bar{n}_0)} = \left(\prod_{k=0}^{p} \frac{\bar{q}(\bar{g}_k, \bar{g}_k'; \bar{m}_k)}{\bar{q}(\bar{g}_k', \bar{g}_k; \bar{m}_k)}\right) \frac{\bar{q}(\bar{g}, \bar{g}'; \bar{m})}{\bar{q}(\bar{g}', \bar{g}; \bar{m})}$$

$$\frac{\pi(\bar{n})}{\pi(\bar{n}_0)} = \left(\prod_{k=0}^{p} \frac{\bar{q}(\bar{g}_k, \bar{g}_k'; \bar{m}_k)}{\bar{q}(\bar{g}_k', \bar{g}_k; \bar{m}_k)}\right).$$

As a consequence we find

$$\pi(\bar{m} + \bar{g})\bar{q}(\bar{g}, \bar{g}'; \bar{m}) = \pi(\bar{m} + \bar{g}')\bar{q}(\bar{g}', \bar{g}; \bar{m})$$

implying that the $\bar{q}$-process is strongly reversible with $\bar{\pi} = \pi$.
"only if" If π satisfies GLB then by Theorem 2.16 the $\bar{q}$-process is strongly reversible with $\bar{\pi} = \pi$. Now consider some path of the given form, then from (2.8) we find

$$\prod_{k=0}^{p} \frac{\bar{q}(\bar{g}_k, \bar{g}_k'; \bar{m}_k)}{\bar{q}(\bar{g}_k', \bar{g}_k; \bar{m}_k)} = \prod_{k=0}^{p} \frac{\bar{\pi}(\bar{m}_k + \bar{g}_k')}{\bar{\pi}(\bar{m}_k + \bar{g}_k)} = \prod_{k=0}^{p} \frac{\pi(\bar{m}_{k+1} + \bar{g}_{k+1})}{\pi(\bar{m}_k + \bar{g}_k)} = \frac{\pi(\bar{n})}{\pi(\bar{n}_0)}.$$

□

Remark 2.19 (Path (2.12)) Note that the transitions in each quotient of (2.11) occur inside a local subset, but that from one quotient to another we may move from one local subset to another (also see Remark 2.7). Further note that the left hand side of (2.11) necessarily has tc be independent of the path. □

Remark 2.20 (Verification of (2.11)) As the local solutions can often be obtained explicitly, the invariance condition (2.11) can in principle be checked by enumerating all possible trajectories. In various practical situations however it turns out that only a small number of "basic" trajectories or cycles need to be checked as based upon regular structures of the underlying Markov chain. Even more efficient, the local solutions often suggest a form of $\pi(\cdot)$ as based upon the required ratios (2.10). By simply checking (2.4) or (2.6) for this suggested form, GLB and an explicit form of $\pi(\cdot)$ can so be verified directly. The examples in section 4 have all been verified directly in this manner. □

Remark 2.21 (Product form) In analogy with the single transition case (cf. [8]) it will be illustrated in the next sections that the construction (2.11) and thus GLB usually leads to product form type results. □

Remark 2.22 (Non-reversible routing) Note that the original Markov chain is not required to be reversible. For example, routing probabilities p_{ij} from one station to another can be non-reversible, as will be illustrated in section 3.3 and section 4. □

Remark 2.23 (Blocking of transitions) If $x(\bar{g};\bar{m})$ is a solution of (2.6) then it is a solution of a subset of (2.6) also. Therefore, if a certain blocking of transitions comes down to the exclusion of one or more equations of (2.6) we may first try to find a π that satisfies (2.4) as based upon the equations (2.6) for the system without blocking and then conclude that the same solution $x(\bar{g};\bar{m})$ remains valid also for the system with the blocking. Conversely, any solution π that satisfies the GLB equations (2.4) remains valid if we exclude any arbitrary subsets of (2.6), i.e. if we prohibit these transitions to occur. The following three examples make this more precise. If we have a solution $x(\bar{g};\bar{m})$ of (2.6) where all transitions are possible then in each of the following modifications this solution remains valid. Explicit examples will be given later on in section 3.3.4 and section 4.

- For a given $\bar{m}_0$ we can set

  $$q(\bar{g},\bar{g}';\bar{m}_0) = 0$$

  for all $\bar{g},\bar{g}'$. Roughly speaking, the vector $\bar{m}_0$ is not allowed to remain unchanged if no other jobs remain unchanged.

- For a given $\bar{g}_0$ we can set

  $$q(\bar{g}_0,\bar{g}';\bar{m}) = q(\bar{g},\bar{g}_0;\bar{m}) = 0$$

  for all $\bar{g},\bar{g}',\bar{m}$. Roughly speaking, a group $\bar{g}_0$ is not allowed to take part of movements.

- For a given number, say G, we can set

  $$q(\bar{g},\bar{g}';\bar{m}) = 0$$

  whenever $\sum_{i=1}^{N} g_i = G$ or $\sum_{i=1}^{N} g_i' = G$ so that all equations in which $\sum_{i=1}^{N}(n_i - m_i) = G$ or $\sum_{i=1}^{N}(n_i' - m_i) = G$ can be removed from (2.6). In words that is, batch movements of a total of exactly G departures and/or G arrivals are not allowed.

3 Network applications

In Theorem 2.16 all information about the stationary distribution π is enclosed in the stationary distribution $\bar{\pi}$ of the $\bar{q}$-process. With a more explicit form for the transition rates $q(\bar{g},\bar{g}';\bar{m})$ it is possible to extract more explicit product form type results for π. Therefore, in this section we impose a special form for the transition rates which has a natural interpretation in queueing networks. With this special form the complexity of the local equations (2.6) is reduced and leads to factorising results (see (3.8)) for the stationary distribution π with as special application product form results. To this end, assume that the transition rates can be decomposed in

$$q(\bar{g},\bar{g}';\bar{m}) = f(\bar{g},\bar{m}+\bar{g})p(\bar{g},\bar{g}';\bar{m}). \tag{3.1}$$

Here $f(\cdot)$ represents the service characteristics, i.e. protocols and speeds at the stations and $p(\cdot)$ represents the routing characteristics, i.e. routing and blocking probabilities of jobs, both up to a possible numerical factor as shown in Remark 3.1 below. Note that this decomposition is not required in the general setting of section 2.

In the first part of this section we show that for a special form of the service characteristics $f(\cdot)$ the stationary distribution π is a product of a service and a routing factor. In the second part we give some general examples of the special service form required. Finally, in the third part of this section we consider the routing part of the transition rates and give some general examples for $p(\cdot)$.

3.1 Decomposition theorem

Suppose that for arbitrary but given functions $\psi(\cdot)$ and $\phi(\cdot)$

$$f(\bar{g},\bar{n}) = \frac{\psi(\bar{n}-\bar{g})}{\phi(\bar{n})} \tag{3.2}$$

where the function $\phi(\cdot)$ is assumed to be strictly positive at V, i.e. for all $\bar{n} \in V$

$$\phi(\bar{n}) > 0.$$

We make no assumptions about the function $\psi(\cdot)$ and thus $\psi(\cdot)$ is allowed to have 0-values.

Remark 3.1 (Decomposition; interpretation) The decomposition (3.1) of the transition rates is valid in many queueing networks. However, the special form (3.2) may not completely describe the service characteristics. In some situations the actual service characteristics may have the form

$$\hat{f}(\bar{g},\bar{n}) = \frac{\psi(\bar{n}-\bar{g})}{\phi(\bar{n})\beta(\bar{g})} \tag{3.3}$$

where $\beta(\bar{g})$ is to be included to take into account different orderings of the jobs in the batch $\bar{g}$. Assume that with this service function $\hat{f}(\cdot)$ and some routing function $\hat{p}(\cdot)$ the transition rates can be presented in the form

$$q(\bar{g},\bar{g}';\bar{m}) = \hat{f}(\bar{g},\bar{m}+\bar{g})\hat{p}(\bar{g},\bar{g}';\bar{m}). \tag{3.4}$$

We can rewrite this as

$$q(\bar{g},\bar{g}';\bar{m}) = f(\bar{g},\bar{m}+\bar{g})p(\bar{g},\bar{g}';\bar{m})$$

by defining

$$\begin{aligned} f(\bar{g},\bar{n}) &= \hat{f}(\bar{g},\bar{n})\beta(\bar{g}) \\ p(\bar{g},\bar{g}';\bar{m}) &= \frac{\hat{p}(\bar{g},\bar{g}';\bar{m})}{\beta(\bar{g})}. \end{aligned} \tag{3.5}$$

The functions $f(\cdot)$ and $p(\cdot)$ are thus to be seen as a service and routing function possibly up to a numerical factor. That is, a decomposition of the form (3.4) with $\hat{f}(\cdot)$ of the form (3.3) can be rewritten in the form (3.1) with $f(\cdot)$ of the form (3.2). If, in the examples, the service characteristics have the form $\hat{f}(\cdot)$ then we use $f(\cdot)$ and say that we include the numerical factors $\beta(\bar{g})$ in the routing part. If $\hat{p}(\cdot)$ represents the routing characteristics then we may use $p(\cdot)$ and say that we include the numerical factors $\beta(\bar{g})$ in the service part. □

Remark 3.2 (References [7], [21]) In both [7] and [21] a decomposition similar to (3.1) is used. In [7] the service characteristics have the form (3.3) and in [21] the service characteristics have the form (3.2). In both references, however, the routing characteristics may depend on $\bar{g},\bar{g}'$ only, i.e. for some function $\lambda(\cdot)$ we must have that $p(\bar{g},\bar{g}';\bar{m}) = \lambda(\bar{g},\bar{g}')$ for all $\bar{m}$. With this restriction blocking phenomena are excluded from the transition rates. □

The following assumption reduces equation (2.6) and the corresponding assumption (A3) to merely the state-dependent routing equations.

(A4) For any fixed $\bar{m}$ for which $V(\bar{m}) \neq \emptyset$ relation (2.5) holds and for $i \in \{1,\ldots,k(\bar{m})\}$ the following system has a unique positive solution $\{y(\bar{g};\bar{m})|\bar{m}+\bar{g} \in V_i(\bar{m})\}$ up to a constant factor.

$$y(\bar{g};\bar{m})\sum_{\bar{g}'\neq\bar{g}} p(\bar{g},\bar{g}';\bar{m}) = \sum_{\bar{g}'\neq\bar{g}} y(\bar{g}';\bar{m})p(\bar{g}',\bar{g};\bar{m}) \qquad \bar{m}+\bar{g} \in V_i(\bar{m}) \tag{3.6}$$

In analogy with the definition of the $\bar{q}$-process we now define a q_y-process at V.

Definition 3.3 **(q_y-process)** *A Markov chain at V with transition rates q_y which satisfy the following relations is called a q_y-process.*

$$\begin{cases} \textit{For any } \bar{m},\ i=1,\ldots,k(\bar{m}) \textit{ and } \bar{m}+\bar{g},\ \bar{m}+\bar{g}' \in V_i(\bar{m}) \\ \qquad \dfrac{q_y(\bar{g},\bar{g}';\bar{m})}{q_y(\bar{g}',\bar{g};\bar{m})} = \dfrac{y(\bar{g}';\bar{m})}{y(\bar{g};\bar{m})} \\ \textit{and otherwise} \\ \qquad q_y(\bar{g},\bar{g}';\bar{m}) = 0. \end{cases} \tag{3.7}$$

Then the following product form result follows.

Theorem 3.4 (Decomposition Theorem) *π satisfies GLB if and only if the q_y-process is strongly reversible at V. Moreover with π_y its stationary distribution we have for all $\bar{n} \in V$ and B a normalizing constant*

$$\pi(\bar{n}) = B\phi(\bar{n})\pi_y(\bar{n}). \tag{3.8}$$

Proof By virtue of Theorem 2.16 we can prove that π satisfies GLB if and only if the q_y-process is strongly reversible by proving that the $\bar{q}$-process is strongly reversible if and only if the q_y-process is strongly reversible. To this end, first note that $q(\bar{g}, \bar{g}'; \bar{m}) = 0$ if $f(\bar{g}, \bar{m} + \bar{g}) = 0$, i.e. if $\psi(\bar{m}) = 0$. Therefore, if $\psi(\bar{m}) = 0$, within $V(\bar{m})$ there are no transitions possible, i.e. $V(\bar{m})$ consists of singletons only and (2.6) is trivially fulfilled. Now assume that $\psi(\bar{m}) > 0$. Then the balance equations (2.6) and (3.6) are equivalent if we substitute

$$x(\bar{g}; \bar{m}) = \frac{\phi(\bar{m} + \bar{g})}{\psi(\bar{m})} y(\bar{g}; \bar{m}) \tag{3.9}$$

for all $\bar{m}$ and $i \in \{1, \ldots, k(\bar{m})\}$ such that $\bar{m} + \bar{g} \in V_i(\bar{m})$.
Then, from (3.7), (3.9), (2.7), (3.2) and substituting the following relation for π_y and $\bar{\pi}$

$$\bar{\pi}(\bar{n}) = B\phi(\bar{n})\pi_y(\bar{n}) \tag{3.10}$$

it follows that

$$\begin{aligned}
\frac{q_y(\bar{g}, \bar{g}'; \bar{m})\pi_y(\bar{m} + \bar{g})}{q_y(\bar{g}', \bar{g}; \bar{m})\pi_y(\bar{m} + \bar{g}')} &= \frac{y(\bar{g}'; \bar{m})\pi_y(\bar{m} + \bar{g})}{y(\bar{g}; \bar{m})\pi_y(\bar{m} + \bar{g}')} \\
&= \frac{x(\bar{g}'; \bar{m})\frac{\psi(\bar{m})}{\phi(\bar{m}+\bar{g}')}\pi_y(\bar{m} + \bar{g})}{x(\bar{g}; \bar{m})\frac{\psi(\bar{m})}{\phi(\bar{m}+\bar{g})}\pi_y(\bar{m} + \bar{g}')} \\
&= \frac{\bar{q}(\bar{g}, \bar{g}'; \bar{m})\frac{\psi(\bar{m})}{\phi(\bar{m}+\bar{g}')}\pi_y(\bar{m} + \bar{g})}{\bar{q}(\bar{g}', \bar{g}; \bar{m})\frac{\psi(\bar{m})}{\phi(\bar{m}+\bar{g})}\pi_y(\bar{m} + \bar{g}')} \\
&= \frac{\bar{q}(\bar{g}, \bar{g}'; \bar{m})\phi(\bar{m} + \bar{g})\pi_y(\bar{m} + \bar{g})}{\bar{q}(\bar{g}', \bar{g}; \bar{m})\phi(\bar{m} + \bar{g}')\pi_y(\bar{m} + \bar{g}')} \\
&= \frac{\bar{q}(\bar{g}, \bar{g}'; \bar{m})\bar{\pi}(\bar{m} + \bar{g})}{\bar{q}(\bar{g}', \bar{g}; \bar{m})\bar{\pi}(\bar{m} + \bar{g}')}.
\end{aligned}$$

Consequently, the $\bar{q}$-process is strongly reversible with stationary distribution $\bar{\pi}$ if and only if the q_y-process is strongly reversible with stationary distribution π_y, where $\bar{\pi}$ and π_y are related by (3.10). Theorem 2.16 or rather equation (2.9) concludes the proof. □

Remark 3.5 (Service and routing factorization) For $q(\bar{g}, \bar{g}'; \bar{m})$ of the form (3.1) it is always possible to replace (2.6) by (3.6) by using a relation similar to (3.9) as a definition for $y(\bar{g}; \bar{m})$:

$$x(\bar{g};\bar{m}) = y(\bar{g};\bar{m})\frac{1}{f(\bar{g},\bar{m}+\bar{g})}.$$

In general, however, when $f(\cdot)$ does not satisfy (3.2) the stationary distribution will not be a product of a service and a routing factor. This however is up to interpretation as we can always scale the service characteristics to satisfy (3.2) and include the remaining part in the routing characteristics. In other words, generally speaking, the notion of balance per group (GLB), seems to be responsible for a possible factorization of the stationary distribution in a part that mainly covers the service characteristics and a part that mainly covers the routing characteristics. Similar service and routing decompositions for the single transition case (cf. [8], [20]) have hereby been extended to multiple transitions. Particularly, as will be illustrated in section 3.2.1, the more classical recognized factorization to individual stations may come out when $f(\cdot)$ itself has a factorizing form. □

Remark 3.6 ($\psi \neq \phi$) Note that the function $\psi(\bar{m})$ drops out in the proof above. This is a direct consequence of the notion of GLB since we consider only the local subsets $V_i(\bar{m})$ for each fixed $\bar{m}$ so that $\psi(\bar{m})$ is a constant at these local subsets. A similar observation has recently been reported for the single transition case (cf. [20], [22]) and for the multiple transition case (cf. [7], [21]) as an extension of the more "standard" assumption that $\psi = \phi$ (cf. [11], [25], [26]). The form (3.8) for the stationary distribution is a direct consequence of the notion of GLB. □

First, in section 3.2 below, we concentrate on merely the service characteristics $f(\cdot)$ satisfying (3.2), which yield the $\phi(\cdot)$ part in (3.8). Next, in section 3.3, we focus upon general examples for $p(\cdot)$ which yield the $\pi_y(\cdot)$ part in (3.8) By combining any service example from section 3.2 with any routing example from section 3.3 we obtain the product form result (3.8) with $\phi(\cdot)$ and $\pi_y(\cdot)$ specified.

3.2 Service form examples

In the first set of examples (section 3.2.1) the service speed at a station depends on the number of jobs at that station only. Various forms related to discrete-time settings are presented. In the second set of examples (section 3.2.2) the stations are grouped into clusters of stations. The service speed at a station is allowed to depend on the total number of jobs within the clusters.

3.2.1 Station dependent servicing

In the examples below we assume that $f(\cdot)$ has the station factorizing form

$$f(\bar{g},\bar{n}) = \prod_{i=1}^{N} f_i(g_i,n_i) = \prod_{i=1}^{N} \frac{\psi_i(n_i-g_i)}{\phi_i(n_i)}.$$

Here $f_i(\cdot)$ represents the service characteristics at station i and is allowed to depend on the number of jobs present at station i only. Throughout the numerical factor involved is

$$\beta(\bar{g}) = \prod_{i=1}^{N} g_i! \,.$$

In order to show the mathematical unification of the above form, below we present four examples of more or less the same form but with different substitutions for ψ and ϕ required and possible interpretations for both continuous- and discrete-time queueing networks.
The examples a. and d. are well-known for discrete-time queueing networks (cf. [16], [24]). The examples b. and c. are believed to be new.
To illustrate the possible physical behaviour of service protocols which lead to these forms we include an interpretation. This interpretation however may not completely cover the implications, and is far from unique. It illustrates however the complexities which do not exist in the standard single change case.

a. Service upon jobs selected in advance
Assume that the service characteristics at station i have the form analogously to the form used in [16], [24]

$$\hat{f}_i(g_i, n_i) = c_i(n_i)\frac{\alpha_i(n_i)\alpha_i(n_i-1)\cdots\alpha_i(n_i-g_i+1)}{g_i!}.$$

By including the numerical factors $\beta(\bar{g})$ in the routing part, as per (3.5), the service function $f(\cdot)$ satisfies (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{N}\prod_{k=1}^{m_i}\frac{1}{\alpha_i(k)}$$
$$\phi(\bar{n}) = \prod_{i=1}^{N}\frac{1}{c_i(n_i)}\prod_{k=1}^{n_i}\frac{1}{\alpha_i(k)}.$$

Interpretation 3.7 The service characteristics above may be interpreted in the following way. Suppose jobs move around in the station randomly at very high speed. At exponential times with rate $c_i(n_i)$ depending on the number of jobs present a server (e.g. a carrousel in manufacturing or time-frame in token ring protocols) passes by to service a selected number of jobs. If k jobs have not been considered for selection yet then with probability $\alpha_i(k)$ a job arriving at the server is serviced. When the first g_i jobs that have arrived at the server are considered for selection then the server leaves the station immediately. Then the service characteristics have the form stated above. □

b. Service on a number of jobs selected while servicing
Assume that the service characteristics at station i have the form

$$\hat{f}_i(g_i, n_i) = c_i(n_i)\frac{\alpha_i(n_i)\cdots\alpha_i(n_i-g_i+1)}{g_i!}\frac{[1-\alpha_i(n_i-g_i)]\cdots[1-\alpha_i(1)]}{(n_i-g_i)!}.$$

By including the numerical factors $\beta(\bar{g})$ in the routing part, as per (3.5), the service function $f(\cdot)$ satisfies (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{N} \frac{1}{m_i!} \prod_{k=1}^{m_i} \frac{1-\alpha_i(k)}{\alpha_i(k)}$$
$$\phi(\bar{n}) = \prod_{i=1}^{N} \frac{1}{c_i(n_i)} \prod_{k=1}^{n_i} \frac{1}{\alpha_i(k)}.$$

Interpretation 3.8 Reconsider the interpretation given in a. above. The service protocol is changed in the following way. Again, the first g_i jobs arriving at the server are all serviced, as according to the probabilities $\alpha_i(k)$, but only if all remaining $n_i - g_i$ jobs are also considered and not selected for service as according to these probabilities. Otherwise more jobs are serviced. The number g_i is thus determined by these probabilities $\alpha_i(k)$. □

c. Service selection until failure

Assume the service characteristics at station i have the form

$$\hat{f}_i(g_i, n_i) = c_i(n_i) \frac{\alpha_i(n_i) \cdots \alpha_i(n_i - g_i + 1)}{g_i!} \left[1 - \alpha_i(n_i - g_i)\right].$$

By including the numerical factors $\beta(\bar{g})$ in the routing part, as per (3.5), the service function $f(\cdot)$ satisfies (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{N} (1 - \alpha_i(\bar{m})) \prod_{k=1}^{m_i} \frac{1}{\alpha_i(k)}$$
$$\phi(\bar{n}) = \prod_{i=1}^{N} \frac{1}{c_i(n_i)} \prod_{k=1}^{n_i} \frac{1}{\alpha_i(k)}.$$

Interpretation 3.9 Reconsider the interpretation given in a. but now the server will service jobs until failure, i.e. the servicing of jobs will continue until one job is not accepted by the server. If the server leaves the station immediately after a failure the service characteristics have the form stated above. □

d. Service for the whole group

A special case of the service characteristics described in a. and b. above is service for a whole group. Assume that the service characteristics at station i have the form

$$\hat{f}_i(g_i, n_i) = \binom{n_i}{g_i} p_i^{g_i} (1 - p_i)^{n_i - g_i}. \tag{3.11}$$

By including the numerical factors $\beta(\bar{g})$ in the routing part, as per (3.5), the service function $f(\cdot)$ satisfies (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{N} \frac{1}{m_i!} \left(\frac{1-p_i}{p_i}\right)^{m_i}$$
$$\phi(\bar{n}) = \prod_{i=1}^{N} \frac{1}{n_i!} \left(\frac{1}{p_i}\right)^{n_i}.$$

Interpretation 3.10 The service characteristics above correspond to a discrete-time infinite server queue with geometrical services with probability of success p_i. □

The service characteristics (3.11) correspond to the following choices for $\alpha_i(k)$ and $c_i(n_i)$ in a. and b. above respectively

$$\begin{array}{llll} \text{a.} & \alpha_i(k) = k\dfrac{p_i}{1-p_i} & c_i(n_i) = (1-p_i)^{n_i} & i = 1,\ldots,N \\ \text{b.} & \alpha_i(k) = p_i & c_i(n_i) = n_i! & i = 1,\ldots,N. \end{array}$$

e. Service delay

Any of the examples a. - d. can be extended to an additional delay factor ω_i for station i as in Example 3.2.2.c. with station i representing cluster i.

Remark 3.11 **($\phi(\cdot)$ part)** Note that the $\phi(\cdot)$ parts in a., b. and c. above are the same. Roughly speaking, $\phi(\cdot)$ contains the service characteristics of the actually serviced jobs only. □

Remark 3.12 **(Combinations)** We may combine stations with service characteristics of the form described in the examples above into one queueing network. In that case the service characteristics $f(\cdot)$ are a product of different types of service characteristics $f_i(\cdot)$. □

3.2.2 Cluster dependent servicing

In this section we suppose that the stations of the queueing network are grouped into disjoint clusters C_i, $i = 1,\ldots,P$ and that the service speed at a station within the cluster is a function of the total number of jobs in that cluster.

More precisely, we introduce the following notation for the number of jobs in a cluster

$$\begin{aligned} t_i &= \sum_{j\in C_i} n_j \quad i = 1,\ldots,P \\ r_i &= \sum_{j\in C_i} g_j \quad i = 1,\ldots,P \end{aligned}$$

i.e. $\bar{t} = (t_1,\ldots,t_P)$ represents the total number of jobs in the clusters and $\bar{r} = (r_1,\ldots,r_P)$ represents the total number of jobs departing from stations in the clusters.

The examples in section 3.2.1 above can easily be generalized to examples for clusters of stations. Also, the examples given for clusters of stations can easily be seen as examples for single stations. To this end, it is sufficient to set the number of stations contained in a cluster equal to one.

The importance of cluster dependent servicing becomes clear when the routing is taken into account also. We may service jobs with a cluster dependent and thus station interdependent servicing, but the routing is still to be considered in a more detailed manner. Further as in example c. we may even have a cluster interdependent servicing.

a. Service upon jobs selected in advance

We now assume a service characteristic analogously to the service characteristics in 3.2.1.a. To this end, assume that jobs within a cluster are picked out one after the other but at exponential picking times. This gives the following service characteristic

$$f(\bar{g},\bar{n}) = f(\bar{r},\bar{t}) = \prod_{i=1}^{P} d_i(t_i)\alpha_i(t_i)\alpha_i(t_i-1)\cdots\alpha_i(t_i-r_i+1)$$

which satisfies (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{P} \prod_{k=1}^{\sum_{j\in C_i} m_j} \frac{1}{\alpha_i(k)}$$
$$\phi(\bar{n}) = \prod_{i=1}^{P} d_i(\sum_{j\in C_i} n_j) \prod_{k=1}^{\sum_{j\in C_i} n_j} \frac{1}{\alpha_i(k)}.$$

b. Service for the whole group

As a special application, in analogy with 3.2.1.d. we consider the case in which jobs are independently selected to be serviced with probability p_i, $i = 1, \ldots, P$. Selected jobs are simultaneously serviced and released immediately after completing service. The service characteristics have the form

$$f(\bar{r},\bar{t}) = \prod_{i=1}^{P} \frac{t_i!}{(t_i-r_i)!} p_i^{r_i}(1-p_i)^{t_i-r_i}$$

which satisfies (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{P} \frac{1}{\left(\sum_{j\in C_i} m_j\right)!} \left(\frac{1-p_i}{p_i}\right)^{\sum_{j\in C_i} m_j}$$
$$\phi(\bar{n}) = \prod_{i=1}^{P} \frac{1}{\left(\sum_{j\in C_i} n_j\right)!} \left(\frac{1}{p_i}\right)^{\sum_{j\in C_i} n_j}.$$

c. Service delay

Suppose we wish that the number of jobs in cluster 1 does not exceed some fixed number T_1 too much. This can be achieved by slowing down service at the other clusters with a factor ω_i at cluster i, $i = 2, \ldots, N$ when $t_1 > T_1$, where it is assumed that $\omega_i \neq 0$ for all i. We thus obtain a cluster interdependent servicing. More precisely, we can have that

$$f(\bar{r},\bar{t}) = \prod_{i=1}^{P} f_i(\bar{r},\bar{t})$$

where $f_i(\cdot)$ has the form

$$f_i(\bar{r},\bar{t}) = \begin{cases} \dfrac{\psi_i(t_i - r_i)}{\phi_i(t_i)} & \text{if } i = 1 \text{ or } i = 2,\dots,P \text{ and } t_1 \le T_1 \\ \dfrac{\psi_i(t_i - r_i)}{\omega_i \phi_i(t_i)} & \text{if } i = 2,\dots,P \text{ and } t_1 > T_1. \end{cases}$$

Then the service characteristics have the form (3.2) if we set

$$\psi(\bar{m}) = \prod_{i=1}^{P} \psi_i(\sum_{j\in C_i} m_j)$$

$$\phi(\bar{n}) = \left[\prod_{i=2}^{P}\omega_i - \left(\prod_{i=2}^{P}\omega_i - 1\right)\mathbf{1}(\sum_{j\in C_1} n_j \le T_1)\right]\prod_{i=1}^{P}\phi_i(\sum_{j\in C_i} n_j)$$

with $\mathbf{1}(A)$ denoting the indicator of an event A, i.e. $\mathbf{1}(A) = 1$ if event A is satisfied and $\mathbf{1}(A) = 0$ otherwise.

3.3 Routing form examples

Throughout this section we assume that the routing characteristics have the form

$$p(\bar{g},\bar{g}';\bar{m}) = \lambda(\bar{g},\bar{g}')b(\bar{g},\bar{g}';\bar{m}) \tag{3.12}$$

where $\lambda(\cdot)$ represents the state-independent multiple job routing probabilities and $b(\cdot)$ is a blocking function.
In the first and second example we assume that $b(\cdot) = 1$, i.e. that no blocking occurs. In the third and fourth example we illustrate two possible types of blocking that can be modeled in our description.

3.3.1 No blocking; batch routing

Suppose that there exists a positive solution $y(\bar{g})$ of the batch traffic equations

$$y(\bar{g})\sum_{\bar{g}'\neq\bar{g}}\lambda(\bar{g},\bar{g}') = \sum_{\bar{g}'\neq\bar{g}} y(\bar{g}')\lambda(\bar{g}',\bar{g}) \tag{3.13}$$

and suppose that no blocking occurs, i.e. that for all $\bar{g},\bar{g}',\bar{m}$

$$b(\bar{g},\bar{g}';\bar{m}) = 1.$$

Then, for all $\bar{m}$ such that $\bar{m}+\bar{g}\in V$ a solution of (3.6) is given by

$$y(\bar{g};\bar{m}) = y(\bar{g}).$$

If there exists a function $\Gamma(\cdot)$ such that for all $\bar{m}$, all i and all $\bar{m}+\bar{g},\bar{m}+\bar{g}'\in V_i(\bar{m})$

$$\frac{\Gamma(\bar{m}+\bar{g})}{\Gamma(\bar{m}+\bar{g}')} = \frac{y(\bar{g};\bar{m})}{y(\bar{g}';\bar{m})}, \tag{3.14}$$

then the $\pi_y(\cdot)$ part of (3.8) is given by

$$\pi_y(\bar{n}) = \Gamma(\bar{n})$$

for all $\bar{n}\in V$.

3.3.2 No blocking; independent routing

We now give a special form for the routing probabilities $\lambda(\bar{g}, \bar{g}')$ for which there exists a function $\Gamma(\bar{m} + \bar{g})$ that satisfies (3.14).
Suppose jobs route among the stations independent of the state of the queueing network and independent of the other routing jobs. Consider a transition from state $\bar{n}$ to state $\bar{n}'$ due to $\bar{g}$ jobs leaving and $\bar{g}'$ jobs entering the stations. The transition rate for this transition has the form (3.12) with $b(\cdot) = 1$.
The state-independent batch routing probabilities for the closed queueing network have the form

$$\lambda(\bar{g}, \bar{g}') = \sum_{\left\{ \begin{array}{l} g_{ij},\ i=1,\ldots,N,\ j=1,\ldots,N,\ :\ g_{ij} \geq 0,\ g_{ij}=0 \text{ if } p_{ij}=0, \\ \sum_{j=1}^{N} g_{ij} = g_i,\ i=1,\ldots,N,\ \sum_{i=1}^{N} g_{ij} = g_j',\ j=1,\ldots,N \end{array} \right\}} \prod_{i=1}^{N} \binom{g_i}{g_{i1},\ldots,g_{iN}} \prod_{j=1}^{N} p_{ij}^{g_{ij}}. \tag{3.15}$$

where p_{ij} represents the routing probability for a single job to route from station i to station j and g_{ij} represents the number of jobs that route from station i to station j.
If there exists a positive solution γ_j, $j = 1, \ldots, N$ of the routing equations

$$\gamma_j = \sum_{i=1}^{N} \gamma_i p_{ij} \qquad j = 1, \ldots, N$$

then a solution of the batch traffic equations (3.13). is given by

$$y(\bar{g}) = \prod_{i=1}^{N} \frac{\gamma_i^{g_i}}{g_i!}. \tag{3.16}$$

If we include the numerical factors $\beta(\bar{g}) = \prod_{i=1}^{N} g_i!$ appearing in (3.15) in the service part, then these factors do not appear in $y(\cdot)$ above, and the function

$$\Gamma(\bar{m} + \bar{g}) = \prod_{i=1}^{N} \gamma_i^{m_i + g_i}$$

satisfies (3.14). The $\pi_y(\cdot)$ part of (3.8) is then given by

$$\pi_y(\bar{n}) = \prod_{i=1}^{N} \gamma_i^{n_i} \tag{3.17}$$

for all $\bar{n} \in V$.
We may model an open queueing network also. To this end, assume that with probability $P(g_0)$ a batch consisting of g_0 jobs enters the network. After arrival in the network jobs are independently routed to the stations. With probability p_{0i} an

arriving job is routed to station i. Then the open queueing network form of (3.15) is given by

$$\lambda(\bar{g},\bar{g}') = \sum_{g_0=0}^{\infty}\sum_{g_0'=0}^{\infty} P(g_0) \quad \times$$

$$\times \sum_{\left\{\substack{g_{ij},\ i=0,\ldots,N,\ j=0,\ldots,N,\ :\ g_{ij}\geq 0,\ g_{ij}=0 \text{ if } p_{ij}=0,\ g_{00}=0, \\ \sum_{j=0}^{N} g_{ij}=g_i,\ i=0,\ldots,N,\ \sum_{i=0}^{N} g_{ij}=g_j',j=0,\ldots,N}\right\}} \prod_{i=0}^{N}\binom{g_i}{g_{i0},\ldots,g_{iN}}\prod_{j=0}^{N} p_{ij}^{g_{ij}}. \tag{3.18}$$

Assume arrivals into the network occur due to a Poisson process with parameter γ_0, i.e.

$$P(g_0) = \frac{\gamma_0^{g_0}}{g_0!}\,\mathrm{e}^{-\gamma_0},$$

then if there exists a positive solution $\gamma_j,\ j=1,\ldots,N$ of the routing equations

$$\gamma_j = \gamma_0 p_{0j} + \sum_{i=1}^{N}\gamma_i p_{ij} \qquad j=1,\ldots,N$$

then (3.16) is a solution of the batch traffic equations. If we include the numerical factors $\beta(\bar{g}) = \prod_{i=1}^{N} g_i!$ in the service part, then the π_y part of (3.8) is given by (3.17). We may use different forms for $P(\cdot)$ yielding different forms for the π_y part (also see section 4.1).

Remark 3.13 (Multiple job routing probabilities) In various papers on queueing networks with multiple job routing it is assumed that jobs route independent of the other routing jobs and independent of the state of the network (cf. [16], [21], [24]). However, the explicit form for the state-independent multiple job routing probabilities $\lambda(\cdot)$ has not been reported yet. □

3.3.3 Upper limit blocking; reversible routing

Suppose in each station a maximum number of jobs is allowed, say $n_i \leq M_i$ at station i, and that the routing characteristics $p(\bar{g},\bar{g}';\bar{m})$ have the form (3.12) with

$$b(\bar{g},\bar{g}';\bar{m}) = \mathbf{1}(\bar{m}+\bar{g}' \leq \bar{M}).$$

In words that is, jobs route in groups but are also blocked as a total group. When a group is blocked the total group returns to their originating stations.
Now assume that the state-independent batch routing is reversible, i.e. that the solution $y(\cdot)$ of the batch traffic equations (3.13) satisfies

$$y(\bar{g})\lambda(\bar{g},\bar{g}') = y(\bar{g}')\lambda(\bar{g}',\bar{g}).$$

Then

$$y(\bar{g};\bar{m}) = y(\bar{g})\mathbf{1}(\bar{m}+\bar{g} \leq \bar{M})$$

is a solution of (3.6) and with $\Gamma(\cdot)$ satisfying (3.14) the π_y part of (3.8) is given by

$$\pi_y(\bar{n}) = \Gamma(\bar{n}) \qquad (\bar{n} \leq \bar{M}).$$

Remark 3.14 (Reference [2]) Reference [2] is restricted to reversible processes only, but allows general symmetrical blocking functions. For the present setting that is with $b(\bar{g},\bar{g}';\bar{m})$ symmetrical in $\bar{g},\bar{g}'$, i.e. for all $\bar{g},\bar{g}'$ such that $\bar{m}+\bar{g},\bar{m}+\bar{g}' \in V$. Note that the indicator blocking presented here is indeed a symmetrical function restricted to V. Under the reversible routing condition given here, similarly to [2] we can extend the above and provide an explicit expression for π_y for any such blocking function. □

3.3.4 Minimal workload blocking; non-reversible routing

Suppose that in each transition a minimum number of jobs must remain in the stations, say M_i at station i. Then a transition from state $\bar{n}$ to state $\bar{n}'$ due to $\bar{g}$ jobs leaving and $\bar{g}'$ jobs entering the stations is blocked when $n_i - g_i < M_i$ for at least one i. In this case all jobs must remain in their originating stations.
Suppose that $p(\cdot)$ has the form (3.12), then the blocking function has the form

$$b(\bar{g},\bar{g}';\bar{m}) = \mathbf{1}(\bar{m} \geq \bar{M}).$$

If $y(\cdot)$ satisfies the batch routing equations (3.13), then for all $\bar{m}$ and all i such that $\bar{m}+\bar{g} \in V_i(\bar{m})$ a solution of (3.6) is given by

$$y(\bar{g};\bar{m}) = y(\bar{g})\mathbf{1}(\bar{m} \geq \bar{M}).$$

If there exists a function $\Gamma(\cdot)$ such that for all $\bar{m}$, i and $\bar{m}+\bar{g},\bar{m}+\bar{g}' \in V_i(\bar{m})$ we have that

$$\frac{\Gamma(\bar{m}+\bar{g})}{\Gamma(\bar{m}+\bar{g}')} = \frac{y(\bar{g})}{y(\bar{g}')}$$

then the π_y part of (3.8) is given by

$$\pi_y(\bar{n}) = \Gamma(\bar{n}) \qquad (\bar{n} \geq \bar{M}).$$

Remark 3.15 (Exclusion of a subset of (3.6)) In fact, the example above can be seen as an application of Remark 2.23 as we exclude from (3.6) all equations with $\bar{m} < \bar{M}$. □

4 Specific examples

In principle we can combine any example from section 3.2 with any example from section 3.3 and obtain the stationary distribution

$$\pi(\bar{n}) = B\phi(\bar{n})\pi_y(\bar{n})$$

with $\phi(\cdot)$ specified in section 3.2 and $\pi_y(\cdot)$ specified in section 3.3.
Below, however, we wish to highlight some specific examples of interest in themselves.

4.1 Continuous-time analogues for discrete-time queueing networks

The most practical examples of queueing networks with multiple transitions that have been reported in the literature are discrete-time queueing networks. Most notably in this respect are [16], [24]. Below we will show that their results can rather directly be concluded from our general framework. An interpretation of the discrepancy between discrete- and continuous-time stationary distributions is hereby provided. The other related papers [6], [7], [19], [21] will be addressed briefly in section 4.1.3.

In discrete-time queueing networks the time axis is segmented into time intervals or slots of fixed length. In each time interval the probability of servicing a number of jobs and the probability for the arrival of a number of jobs is defined. In general, these probabilities depend on the number of jobs that are present during a time slot.
Regarding the number of jobs that can be serviced during a time slot there are two conventions (cf. [16]):

(A) *Late arrivals:* a job that arrives at a station during a time slot can not be serviced in the same slot;

(B) *Early arrivals:* a job that arrives at a station during a time slot has a non zero probability to be serviced in the same slot.

In both [16] and [24] it is shown that a discrete-time queueing network with early arrivals has a stationary distribution similar to a standard type continuous-time Jackson network with single changes.
Below we will show that also for continuous-time queueing networks with multiple job transitions this observation remains valid. We note, however, that the physical behaviour of this continuous-time queueing network may differ from that of the discrete-time queueing network. Also, the stationary distributions are slightly different due to the different corresponding discrete- and continuous-time global balance equations. More precisely, as will appear, a continuous-time queueing network with transition rates equal to the transition probabilities of a discrete-time queueing network does not give exactly the same stationary distribution as the discrete-time queueing network.

4.1.1 Walrand's discrete-time queueing network

In [24] a discrete-time queueing network with early arrivals is studied. The arrivals into the network are assumed to be Poisson. In any given time slot station i will serve g_i out of n_i jobs with probability $S_i(g_i, n_i)$ given by

$$\begin{aligned} S_i(0,n_i) &= c_i(n_i) \\ S_i(g_i,n_i) &= c_i(n_i)\frac{\alpha_i(n_i)\alpha_i(n_i-1)\cdots\alpha_i(n_i-g_i+1)}{g_i!} \quad 0 < g_i \leq n_i \end{aligned}$$

where $\alpha_i(0) = 1, \ \alpha_i(k) > 0$ for $k > 0$ and $c_i(n_i)$ such that

$$\sum_{k=0}^{n_i} S_i(k, n_i) = 1 \quad i = 1, \ldots, N.$$

After completing service the jobs are independently routed among the arcs of the queueing network. Then the routing characteristics have the form (3.18).

The continuous-time queueing network with the transition rates equal to the transition probabilities above is a combination of Example 3.2.1.a. and Example 3.3.2. Notice that the binomial factors $\beta(\bar{g}) = \prod_{i=1}^{N} g_i!$ thus appear both in the service characteristics and in the routing characteristics as is a direct example of the redefinition (3.5) stated in Remark 3.1.

The stationary distribution at V becomes

$$\pi(\bar{n}) = B \prod_{i=1}^{N} \frac{\gamma_i^{n_i}}{c_i(n_i)} \prod_{k=1}^{n_i} \frac{1}{\alpha_i(k)}. \tag{4.1}$$

Interpretation 4.1 (Difference) In contrast with the stationary distribution obtained in [24] for the discrete-time queueing network in (4.1) the factors $c_i(n_i)^{-1}$ appear. To give some insight in this difference consider the queueing network in which the arrival process is stopped. In the discrete-time queueing network at station i the expected number of slots until jobs are served when n_i jobs are present is given by

$$E(\# \text{ slots }) = \sum_{k=0}^{\infty} k \, [c_i(n_i)]^k \, [1 - c_i(n_i)] = \frac{c_i(n_i)}{1 - c_i(n_i)}.$$

In the continuous-time queueing network at station i the expected time until a service completion is given by

$$E(\text{ time }) = \frac{1}{1 - c_i(n_i)}.$$

As the stationary distribution represents the fraction of time spent in each state, the ratio of the number of slots until service in the discrete-time model and the time until service in the continuous-time model equals the ratio of the discrete-time and the continuous-time stationary distribution. With π_d, π_c the discrete- and continuous-time stationary distribution of queue i respectively, we thus obtain

$$\frac{\pi_c}{\pi_d} = \frac{E(\text{ time })}{E(\# \text{ slots })} = \frac{1}{c_i(n_i)}.$$

For example, think of

$$c_i(n_i) = 1 - n_i \mu \Delta$$

as corresponding to a discrete-time analog of an infinit server queue with time slots of length Δ and service intensity μ per unit of time. We conclude that this discrepancy vanishes as $\Delta \to 0$. $\square$

4.1.2 Pujolle's network of "extended Bernoulli queues"

In [16] discrete-time queueing networks of "extended Bernoulli queues" with early or late arrivals are studied. However, the discrete-time queueing network with both multiple arrivals and multiple departures is assumed to have early arrivals. The arrivals into the network occur due to a generalized Bernoulli process. The probability of g_0 arrivals in a given time slot is then given by

$$P(g_0) = \left(\frac{p}{1-p}\right)^{g_0} \left(\frac{1-2p}{1-p}\right) \qquad (p < \tfrac{1}{2}).$$

The service is assumed to be a generalized Bernoulli process, also. The probability of servicing g_i out of n_i jobs present at station i in a given time slot is then given by

$$R_i(g_i, n_i) = \frac{p_i^{g_i}(1-p_i)^{n_i-g_i}}{(1-p_i)^{n_i} + (1-p_i)^{n_i-1}p_i + \cdots + p_i^{n_i}} \qquad (p_i < \tfrac{1}{2}).$$

It is assumed that jobs route among the arcs of the network independent of the other routing jobs and independent of the state of the network.
The continuous-time queueing network with transition rates equal to the transition probabilities above is a combination of Example 3.2.1.a and a slight modification of Example 3.3.2.
Then, the π_y part of the stationary distribution is given by

$$\pi_y(\bar{n}) = \prod_{i=1}^{N} \left(\frac{p\gamma_i}{1-p}\right)^{n_i}.$$

The stationary distribution is given by

$$\pi(\bar{n}) = B \prod_{i=1}^{N} \frac{1}{c_i(n_i)} \left[\frac{p(1-p_i)\gamma_i}{(1-p)p_i}\right]^{n_i}$$

with

$$c_i(n_i) = \frac{(1-p_i)^{n_i}}{(1-p_i)^{n_i} + (1-p_i)^{n_i-1}p_i + \cdots + p_i^{n_i}}$$

and γ_i the solution of the routing equations.
Again, in contrast with the discrete-time stationary distribution the factors $c_i(n_i)^{-1}$ appear in the continuous-time stationary distribution.

4.1.3 Further references

In [6], [19] discrete-time queueing networks with so-called doubly stochastic queues are studied. In [6] open networks with geometrical input streams are studied, whereas in [19] a specific computer model is analyzed to show how to deal with closed queueing networks. In both references a transition is allowed only if there is just one job requesting a transition. If more than one job completes service during a time-slot then all these jobs have to restart service. If there is exactly one job completing service the transition is allowed. If the transition is allowed then the job routes among the arcs of the queueing network and is not blocked, i.e. it arrives at some other station of the queueing network independent of the state of the queueing network.

In [7], [21] a framework of continuous-time queueing networks with multiple job transitions is given. These continuous-time models are used to model discrete-time queueing networks. In [21] closed queueing networks are studied. The service characteristics have the form (3.2) and jobs route independent of the state of the queueing network and independent of the other routing jobs, i.e. the routing characteristics have the form given in Example 3.3.2. In [7] closed and open queueing networks are studied. The service characteristics have the form (3.3) and the routing characteristics have the more general form described in Example 3.3.1.

In non of the above references blocking can be modeled. In our general framework we are allowed to include blocking. As is illustrated in section 3.3 we may include various types of blocking in the examples above. In the examples below we illustrate some specific types of blocking that can be modeled in our general framework.

4.2 Cyclic three center model; anticipative blocking

Reconsider the cyclic three center model of Figure 1 but now with M jobs present. At station 3 no more than N_3 jobs are allowed. The state space of this closed queueing network is given by

$$V = \left\{ \bar{n} : \sum_{i=1}^{3} n_i = M, \; n_3 \leq N_3 \right\}.$$

A transition from state $\bar{n}$ to state $\bar{n}'$ occurs due to the following mechanism. First at each station a group g_i, $i = 1, 2, 3$ is selected. Group g_1 and g_2 are allowed to leave station 1 and 2 if each of these groups independently is allowed to enter station 3 after the departure of group g_3 from station 3, i.e. only if

$$\begin{aligned} m_3 + g_1 &\leq N_3 \\ m_3 + g_2 &\leq N_3. \end{aligned}$$

If either g_1 or g_2 is not allowed to enter station 3 all transitions are blocked and at all stations a new group g_i, $i = 1, 2, 3$ is reselected for servicing. In other words, the state of the system effectively remains the same. Otherwise the groups are routed according to the arcs depicted in Figure 1.
The service mechanism described above is natural in discrete-time queueing networks with late arrivals. First at each station within a time-slot a group g_i, $i = 1, 2, 3$ is serviced. At the end of the time-slot the groups are released in the network and select a new station according to the routing rules. If at some place blocking occurs

all groups return to their stations. Otherwise they arrive at a new station at the beginning of the following time-slot.
The transition rates for this process are given by

$$q(\bar{g},\bar{g}';\bar{m}) = f(\bar{g},\bar{m}+\bar{g})\mathbf{1}(m_3+g_i \leq N_3,\ i=1,2,3) \tag{4.2}$$

Remark 4.2 (Stop protocol for single changes) When single changes are allowed only, as an immediate consequence of (4.2) the service at station 1 and station 2 must be stopped if $n_3 = N_3$. □

The q_y-process is defined through the following set of equations

$$y(\bar{g})p(\bar{g},\bar{g}';\bar{m}) = y(\bar{g}'')p(\bar{g}'',\bar{g};\bar{m}) \tag{4.3}$$

where $\bar{g}' = (g_3,g_1,g_2)$, $\bar{g}'' = (g_2,g_3,g_1)$ if $\bar{g} = (g_1,g_2,g_3)$ and

$$p(\bar{g},\bar{g}';\bar{m}) = \mathbf{1}(m_3+g_i \leq N_3,\ i=1,2,3).$$

For all $\bar{g}$, $\bar{m}$ such that $\bar{m}+\bar{g} \in V(\bar{m})$ a solution of (4.3) is given by

$$y(\bar{g};\bar{m}) = y(\bar{g}) = 1.$$

If the service characteristics have the form (3.2) then by Theorem 3.4 it immediately follows that the stationary distribution at V is given by

$$\pi(\bar{n}) = B\phi(\bar{n}).$$

Remark 4.3 (Blocking and GLB) The blocking protocol described above, where also jobs departing from station 1 are blocked is required for the stationary distribution to satisfy GLB. This can be seen directly if we consider the q_y-process. This process needs to be strongly reversible in order to satisfy the conditions of the Decomposition Theorem. Therefore if, in the q_y-process, a group is allowed to route from station 3 to station 1 this group must be allowed to route from station 1 to station 3 in the q_y-process, also. □

4.3 Random non-reversible routing; selective anticipative blocking

Consider the closed queueing network depicted in Figure 3 in which M jobs are present. Suppose at station 6 no more then N_6 jobs are allowed. The state space of this queueing network is

$$V = \left\{\bar{n} : \sum_{i=1}^{7} n_i = M,\ n_6 \leq N_6\right\}.$$

In the previous example we have shown that in a cyclic queueing network, and thus with deterministic routing, jobs are allowed to leave a station if they can be accepted at the constraint station, only. When the routing is random, however, the protocol is to be chosen more complicated in order to guarantee GLB. In some stations, say at station i the service may still continue at arbitrary speed even though $m_6+g_i > N_6$, while at others servicing of groups g_i is to be stopped as soon as $m_6+g_i \leq N_6$. This also illustrates that the single service blocking protocol "stop all stations if the constraint station reaches its maximum" (cf. [23]) is too restrictive when batch routings are involved. Suppose that the routing part of the transition rates has the form (3.13) with $\lambda(\bar{g},\bar{g}')$ of the form (3.15), i.e. a job routes among the arcs independent of the other routing jobs. Suppose the routing equations possess a positive solution γ_i, $i=1,\ldots,7$. Then the following blocking function $b(\bar{g},\bar{g}';\bar{m})$ is sufficient for the stationary distribution to satisfy GLB

$$b(\bar{g},\bar{g}';\bar{m}) = \mathbf{1}(m_6+g_1+g_2 \leq N_6,\ m_6+g_6' \leq N_6).$$

Here the term $m_6+g_6' \leq N_6$ reflects that from station 4 and station 5 no more than N_6-m_6 jobs are allowed to leave and route to station 6. The service at station 5 may continue for jobs routing to station 7.
If the service characteristics have the form (3.2) then the stationary distribution at V is given by

$$\pi(\bar{n}) = B\phi(\bar{n})\prod_{i=1}^{7}\gamma_i^{n_i}.$$

If we split the queueing network in an upper half above the line through station 5 and a lower half below this line, then we may conclude that in the upper half at each level no more than N_6-g_6 jobs are allowed to leave, where the levels consist of station 1 plus station 2 for the first level, station 4 plus the upper half of station 5 for the second level and station 6 for the third level. In the lower half service may continue at an arbitrary speed.

4.4 Clusters of stations

In the examples below we consider clusters of stations with a constraint on the total number of jobs allowed in the cluster. We use the notation already introduced in Example 3.2.2 and suppose that the service characteristics have the form (3.2).
For simplicity we consider cyclic models only, as in general a characterization for the routing without considering the explicit structure of the queueing network or of the clusters when multiple changes are allowed is most complicated.

4.4.1 Anticipative upper limit blocking

Consider the queueing network depicted in Figure 4 where 1 and 2 are single stations and 3 represents a cluster of stations. The routing among 1, 2 and 3 is assumed to be cyclic, i.e. along the arcs, whereas the internal routing in cluster 3 is arbitrary.

Suppose in cluster 3 no more than T_3 jobs are allowed while at station 1 and 2 all arriving jobs are accepted. If M jobs are present in the queueing network then the state space is given by

$$V = \left\{ \bar{n} : \sum_{i=1}^{N} n_i = M, \sum_{j \in C_3} n_j \leq T_3 \right\}.$$

We introduce a blocking protocol similar to that of Example 4.2 stating that at station 1 and 2 jobs are allowed to leave only if they could be accepted at cluster 3 after the departure of serviced jobs from cluster 3.

a. Batch routing

If we assume that when a transition is allowed the jobs route among the stations with batch routing probabilities $\lambda(\bar{g}, \bar{g}')$, the routing characteristics are given by

$$p(\bar{g}, \bar{g}'; \bar{m}) = \lambda(\bar{g}, \bar{g}')\mathbf{1}(t_3 - g_1' + g_2 \leq T_3,\ t_3 - g_1' + g_2' \leq T_3)$$

Assuming that there exists a solution $\Gamma(\bar{m} + \bar{g})$ as introduced in Example 3.3.1 the stationary distribution of the queueing network at V now becomes

$$\pi(\bar{n}) = B\phi(\bar{n})\Gamma(\bar{n}).$$

b. Independent routing

If we assume that jobs route independent of the other routing jobs while at each cluster first a group r_i is selected and then this group is distributed among the stations of the cluster to be serviced there, the transition rates have the form

$$\begin{aligned} p(\bar{g}, \bar{g}'; \bar{m}) &= \left(\prod_{i=1}^{P} \binom{t_i}{r_i} p_i^{r_i} (1 - p_i)^{t_i - r_i} \frac{r_i!}{\prod_{j \in C_i} g_j!} \right) \times \\ &\qquad \times \lambda(\bar{g}, \bar{g}')\mathbf{1}(t_3 - g_1' + g_2 \leq T_3,\ t_3 - g_1' + g_2' \leq T_3) \end{aligned}$$

where $\lambda(\bar{g}, \bar{g}')$ has the form (3.15).
The stationary distribution at V now becomes

$$\pi(\bar{n}) = \prod_{i=1}^{P} \frac{1}{\left(\sum_{j \in C_i} n_j\right)!} \left(\frac{1}{p_i}\right)^{\sum_{j \in C_i} n_j} \prod_{j \in C_i} \prod_{k=1}^{n_j} \gamma_k^{n_k}.$$

The interpretation of including the numerical factors $\beta(\cdot)$ is illustrated here rather nicely since both $\prod_{i=1}^{P} r_i!$ and $\prod_{j=1}^{N} g_j!$ cancel in $p(\bar{g}, \bar{g}'; \bar{m})$ above.

4.4.2 Anticipative minimal workload blocking

Reconsider the queueing network depicted in Figure 4 where 1 and 2 are single stations and 3 represents a cluster of stations. Suppose in cluster 3 a minimal workload must be guaranteed. In Example 3.3.4 we have considered the case in which after departure a minimum number of jobs had to be present. Here we consider the case in which after arrival a minimum number of jobs has to be present. Suppose in cluster 3 at least T_3 jobs have to be present after a transition, then a transition is allowed if

$$t_3 - g_1' + g_2 \geq T_3,\ t_3 - g_1' + g_1 \geq T_3.$$

If M jobs are present in the queueing network then the state space is given by

$$V = \left\{ \bar{n} : \sum_{i=1}^{N} n_i = M, \sum_{j \in C_3} n_j \geq T_3 \right\}.$$

We consider the batch routing case only. Then the routing characteristics are given by

$$p(\bar{g}, \bar{g}'; \bar{m}) = \lambda(\bar{g}, \bar{g}') \mathbf{1}(t_3 - g_1' + g_2 \geq T_3,\ t_3 - g_1' + g_1 \geq T_3).$$

Assuming that there exists a solution $\Gamma(\bar{m} + \bar{g})$ as introduced in Example 3.3.1 the stationary distribution of the queueing network at V is given by

$$\pi(\bar{n}) = B\phi(\bar{n})\Gamma(\bar{n}).$$

5 Conclusion

We have studied a general framework for continuous-time queueing networks with state-dependent multiple job transitions. By using a key notion of group-local-balance we have presented a constructive method for computing the stationary distribution. This method is based upon local solutions of the group-local-balance equations. Explicit expressions for these local solutions are often obtainable. Therefore, a practical method for computing the stationary distribution is thus provided. For a specific, but rather general choice of the service characteristics we have achieved a decomposition of the stationary distribution in a part that mainly covers the service characteristics and a part that mainly covers the routing characteristics. The state-dependent batch routing equations thus obtained allow features as blocking of transitions. The specific choice for the service characteristics allows station-interdependent servicing. This naturally arises when stations are grouped into clusters such as due to a common storage pool or when the service is slowed down when a station or group of stations becomes saturated.

It is shown in the examples that known results on both discrete- and continuous-time queueing networks are unified and extended. To illustrate the possibilities a number of novel product form examples is included. Particularly, examples with non-reversible routing and multiple job transitions with blocking.

The framework presented here is restricted to one type of jobs. The results can be extended to queueing networks with more types of jobs.

References

[1] Baskett, F., K. M. Chandy, R. R. Muntz and F. G. Palacios (1975). "Open, closed and mixed networks of queues with different classes of customers." *J. A. C. M.* Vol. **22** 248-260.

[2] Boucherie, R. J. and N. M. van Dijk (1988). "Spatial birth-death processes with multiple changes and applications to batch service networks and clustering processes." *Research Report. Free University, Amsterdam, The Netherlands.*

[3] Burman, D. Y., J. P. Lehoczky, and Y. Lim (1982). "Insensitivity of blocking probabilities in a circuit switching network." *J. Appl. Prob.* **21** 850-859.

[4] Chandy, K. M., J. H. Howard and D. F. Towsley (1977). "Product form and local balance in queueing networks." *J. A. C. M.* Vol **24** 250-263.

[5] Chandy, K. M. and A. J. Martin (1983). "A characterization of product-form queueing networks." *J. A. C. M.* Vol. **30** 286-299.

[6] Daduna, H. and R. Schassberger (1983). "Networks of queues in discrete time." *Zeitschrift für Operations Research.* **27** 159-175.

[7] Henderson, W. and P. G. Taylor (1988). "Networks of queues with batch arrivals and batch services." *Technical Report. University of Adelaide.*

[8] Hordijk, A. and N. M. van Dijk (1983). "Networks of queues Part I: Job-local-balance and the adjoint process. Part II: General routing and service characteristics." *Lecture notes in Control and Informational Sciences.* Springer-Verlag. Vol. **60** 158-205.

[9] Jackson, J. R. (1957). "Networks of waiting lines." *Operations Research.* **5** 518-521.

[10] Jackson, J. R.(1963). "Jobshop-like queueing systems." *Management Science* **10** 131-142.

[11] Kelly, F. P.(1979). *Reversibility and stochastic networks.* Wiley.

[12] Krzesinsky, A. E.(1987). "Multiclass queueing networks with state-dependent routing." *Performance Evaluation.* **7** 125-143.

[13] Noetzel, A. S.(1979). "A generalized queueing discipline for product form network solutions." *J. A. C. M.* Vol **26** 779-793.

[14] Pittel, B.(1979). "Closed exponential networks of queues with saturation: the Jackson-type stationary distribution and its asymptotic analysis." *Math. Operations Research.* **4** 357-378.

[15] Pujolle, G., J. P. Claude and D. Seret (1985). "A discrete tandem queueing system with a product form solution." *Proc. Intern. Seminar on Computer Networking and Performance Evaluation.* North-Holland, Kyoto. pp. 139-147.

[16] Pujolle, G.(1988). "Discrete-time queueing systems with a product form solution." *MASI Research Report.*

[17] Schassberger, R.(1978). "The insensitivity of stationary probabilities in networks of queues." *Adv. Appl. Prob.* **10** 906-912.

[18] Schassberger, R.(1979). "A definition of discrete product form distributions." *Zeitschrift für Operations Research.* **23** 189-195.

[19] Schassberger, R. and H. Daduna (1983). "A discrete-time technique for solving closed queueing models of computer systems." *Technical Report. Technische Universität Berlin.*

[20] Serfozo, R. F.(1988). "Markovian network processes with system-dependent transition rates."

[21] Taylor, P. G., W. Henderson, C. E. M. Pearce and N. M. van Dijk (1988). "Closed queueing networks with batch services." *Technical Report. University of Adelaide.*

[22] Van Dijk, N. M.(1988). "Product forms for queueing networks with limited clusters." *Research Report. Free University, Amsterdam, The Netherlands.*

[23] Van Dijk, N. M.(1989). ""Stop = Recirculate" for exponential product form queueing networks with departure blocking." *Research Report. Free University, Amsterdam, The Netherlands.*

[24] Walrand, J.(1983). "A discrete-time queueing network." *J. Appl. Prob.* **20** 903-909.

[25] Walrand, J.(1988). *An introduction to queueing networks.* Prentice Hall.

[26] Whittle, P.(1986). *Systems in stochastic equilibrium.* Wiley.

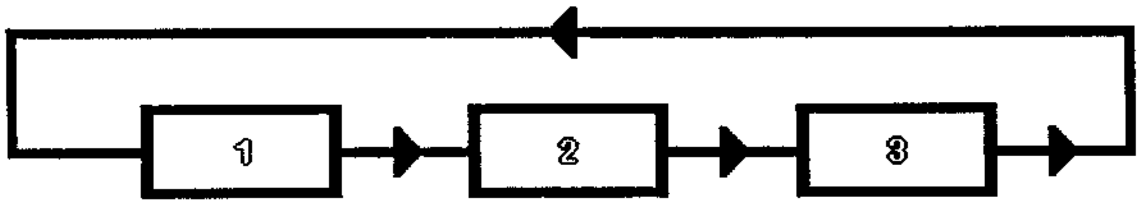

Figure 1. Cyclic three station network

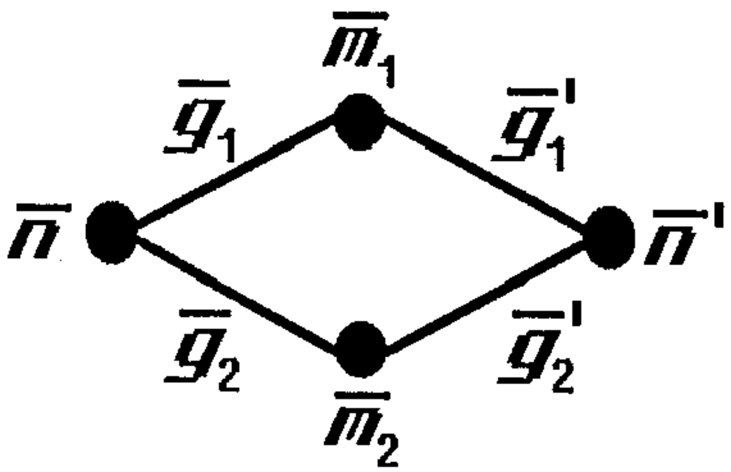

Figure 2. Reversibility

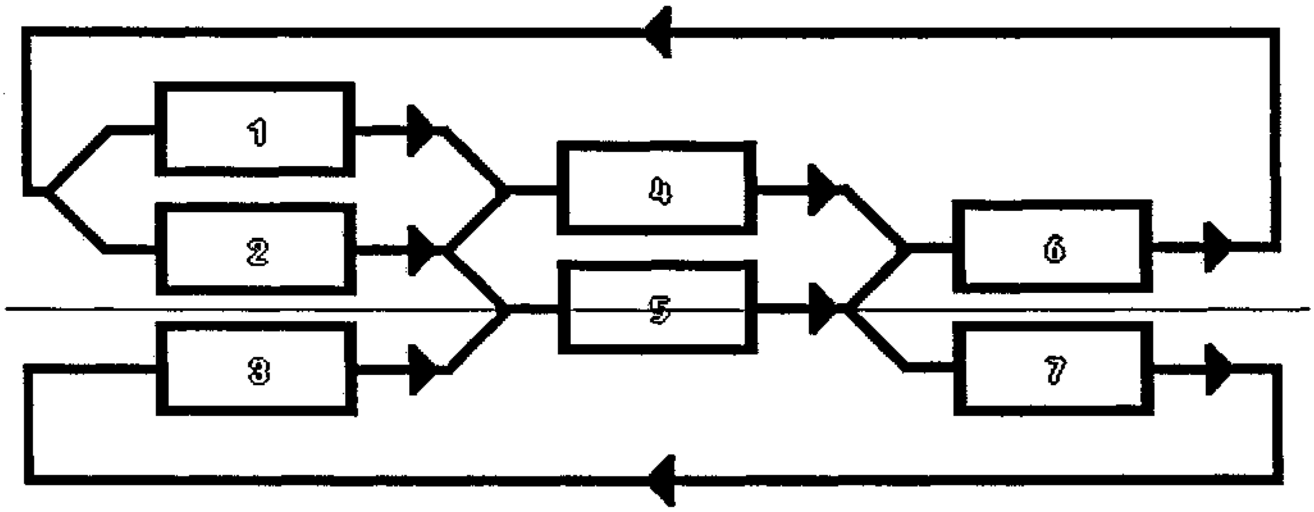

Figure 3. Closed queueing network.

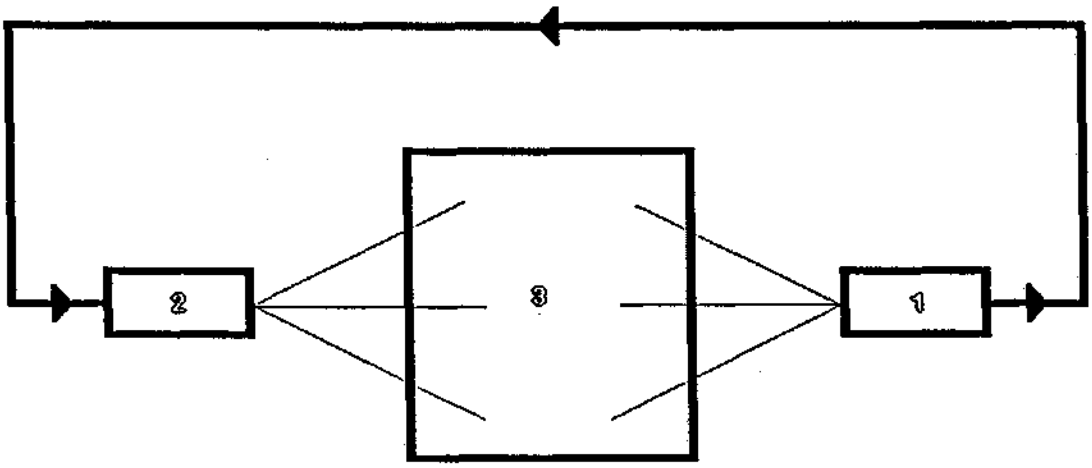

Figure 4. Cyclic queueing network with one cluster.

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